Consistent initial values for DAE systems in circuit simulation

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Abstract

One of the difficulties of the numerical integration methods for differential-algebraic equations (DAEs) is computing consistent initial values before starting the integration, i.e. , calculating values that satisfy the given algebraic constraints as well as the hidden constraints if higher index problems are considered.
This paper presents an approach to calculate consistent initial values for index-2 DAEs starting up from possibly inconsistent ones. Firstly, the idea is exposed for linear DAEs and then it is shown how the results can be applied to those systems arising from modified nodal analysis (MNA) in circuit simulation. This article starts up from [8] and [6]. Several denotations and results we use were introduced there in more detail.

Key words: Consistent initial values; consistent initialization; differential-algebraic equation; DAE; index; circuit simulation; modified nodal analysis; MNA; structural properties.

AMS Subject Classification: 94C05, 65L05.

1 Introduction

Roughly speaking, the problem of determining consistent initial values for differential-algebraic equations can be described as follows. For ordinary differential equations, initial values have to be prescribed for all variables to determine a unique solution. However, differential-algebraic equations consist of differential equations coupled with derivative-free equations, i.e., not all components appear in dynamic form. Indeed, some of them are determined by algebraic constraints. In Section 2 a convenient characterization of consistent initial values is introduced.
One approach to determine consistent initial values is to locate a selection of variables for which we may prescribe initial values and to construct a nonsingular system that provides the values for the remaining ones. In this context, we have to consider two problems:

1. The selection of variables is not arbitrary.

2. The values that are assigned to the selected variables have to be chosen in such a way that the nonlinear system is solvable.

In [6] it was analyzed how to construct a nonsingular system and how to treat (1) for systems arising from circuit simulation by means of modified nodal analysis, while (2) was not discussed. Nevertheless, the practical realization of that approach leads to new insights, provided that the values from (2) are suitably chosen. These results will be presented in the present paper.
To illustrate the approach before considering the special systems arising from circuit simulation, we describe the basic ideas for linear DAEs in Section 3. Then, in Section 4 we introduce the equations of the modified nodal analysis. The structural properties of these systems, which were pointed out in [8] and [6], are summarized in Section 5. The new results are presented in Section 6. Since, in circuit simulation, the operating point is frequently used for starting the integration, it is separately analyzed in Section 6.1. In Section 6.2 the approach is described for a more general case. In practice, the values obtained in the Sections 6.1 and 6.2 can be calculated by solving relatively small linear systems as described in Section 6.3.
Finally, in Section 7 it is shown how this approach may be combined with an initialization strategy that takes into account possible initialization preferences of the user of a simulation package.

2 About consistent initial values for DAEs

We consider differential-algebraic equations, i.e. , equations of the form

$$f(x', x, t) = 0, \tag{2.1}$$

where $\frac{df}{dx'}$ is singular. In this article, $\frac{df}{dx'}$ is assumed to have a constant nullspace. Note that, if we define a projector Q onto $ker \frac{df}{dx'}$ and $P := I - Q$, then equation (2.1) may be written as

$$f(Px', x, t) = 0.$$

Definition 2.1 *A vector $x_0 \in R^m$ is a consistent initial value of (2.1) if there exists a solution of (2.1) that fulfils $x(t_0) = x_0$.*

Taking into account that the singularity of $\frac{df}{dx'}$ implies that (2.1) contains some algebraic equations, a consistent initial value has to fulfil precisely those algebraic equations. Moreover, the differentiation of these algebraic equations may lead to further algebraic equations, called hidden constraints, which a consistent initial value has to fulfil, too. This fact is closely related to the index concept.

Actually, we are also interested in the corresponding values of the derivatives appearing in the DAE, i.e., in the values of Px' if P is defined as $P := I - Q$ for a projector Q onto $ker \frac{df}{dx'}$.

Definition 2.2 *A vector* (x_0, Py_0) *is a consistent initialization of (2.1) if* x_0 *is a consistent initial value and* (x_0, Py_0) *fulfils the equation* $f(Py_0, x_0, t_0) = 0$.

For simplicity, we will first present the approach for linear differential-algebraic equations (DAEs). In the course of the article, it will be shown how it can be extended to those quasi-linear systems obtained by modified nodal analysis (MNA).

3 An overview of the approach for linear DAEs

For a short outline of the main ideas presented in this article, the tractability index and the spaces related to its definition are introduced.

Consider a linear DAE of the form:

$$Ax' + Bx = q(t), \tag{3.1}$$

where A is singular.
For the tractability index we define $N := \ker A$ and $S := \{z : Bz \in \text{im } A\}$.

Definition 3.1 *The DAE (3.1) is called index-1-tractable if the matrix* $A_1 := A + BQ$ *is nonsingular for a constant projector* Q *onto* N.[1]

Remarks:

1. The matrix A_1 is nonsingular if and only if $N \cap S = \{0\}$.
2. The definition does not depend on the choice of the projector Q.

For the definition of the index two we define $N_1 := \ker A_1$ and $S_1 := \{z : BPz \in \text{im } A_1\}$ for $P := (I - Q)$.

Definition 3.2 *The DAE (3.1) is called index-2-tractable if*

1. *it is not index-1-tractable,*
2. $A_2 := A_1 + BPQ_1$ *is nonsingular for a projector* Q_1 *onto* N_1.[2]

Remarks:

1. The matrix A_2 is nonsingular if and only if $N_1 \cap S_1 = \{0\}$.
2. The definition does not depend on the choice of the projector Q_1.

Definition 3.3 *In the following, the canonical projector* $Q_1 := Q_1 A_2^{-1} BP$ *onto* N_1 *along* S_1 *is considered.*

[1] cf. [10].
[2] cf. [10].

In the index-2 case the space $N \cap S$ represents all the components that are determined neither by a differential nor by an algebraic equation. These components can be determined only by inherent differentiation.
Furthermore, for index-2 equations hidden constraints appear if we derive a part of the system's equations. This implies that an initial value has to be chosen in such a way that not only the system's equations, but, additionally, the hidden constraints have to be fulfilled.
An efficient approach[3] to calculate a consistent initialization for index-2 DAEs at t_l consists in solving the system[4]:

$$\begin{aligned} Ay_l + Bx_l &= q(t_l), & (3.2)\\ PP_1(x_l - \alpha) + PQ_1 y_l - PQ_1 A_2^{-1} q'(t_l) + Qy_l &= 0, & (3.3) \end{aligned}$$

for an arbitrary α and $P_1 := I - Q_1$.
Note that $PP_1(x_l - \alpha) = 0$ fixes the dynamic components, $PQ_1 y_l = PQ_1 A_2^{-1} q'(t_l)$ describes the hidden constraints, and $Qy_l = 0$ precisely fixes the values we are not interested in, obtaining a nonsingular system.
The results presented in [8] imply that, for nonlinear circuits, this approach has the disadvantage that the projectors PP_1 and PQ_1 depend on the solution. Nevertheless, in [6] it was pointed out how the method could be reformulated in terms of other constant projectors.

In this article we will suppose that we already know values (x^l, y^l) that fulfil the equations of the system (3.1) at t_l that are not necessarily consistent [5], i.e., which probably do not fulfil the hidden constraints.[6] Without loss of generality, we also assume that $Qy^l = 0$ is fulfilled.
We denote by (x_l, y_l) the consistent value we obtain from (3.2) - (3.3) by setting $\alpha := x^l$ and define:

$$x_l^+ := x_l - x^l, \quad y_l^+ = y_l - y^l. \qquad (3.4)$$

From (3.2) - (3.3) it follows that:

$$\begin{aligned} Ay_l^+ + Bx_l^+ &= 0, & (3.5)\\ PP_1 x_l^+ + PQ_1 y_l^+ + PQ_1 y^l - PQ_1 A_2^{-1} q'(t_l) + Qy_l &= 0, & (3.6) \end{aligned}$$

i. e., we can calculate the value (x_l^+, y_l^+) from that system. Note that (3.6) implies

$$\begin{aligned} PP_1 x_l^+ &= 0, & (3.7)\\ PQ_1 y_l^+ &= PQ_1 A_2^{-1} q'(t_l) - PQ_1 y^l, & (3.8)\\ Qy_l^+ &= 0. & (3.9) \end{aligned}$$

[3] cf. [7]. Note that this approach can also be extended to some nonlinear cases.
[4] We introduce the index l to distinguish the values at an arbitrary time t_l from t_0, which represents the time we start the integration process.
[5] These values may be known from an integration process.
[6] In the following we consider only the index-2 case, because in the index-1 case a value that fulfils the system's equations is automatically consistent.

The task of determining values for (x_l^+, y_l^+) by making use of (3.5) - (3.6) may look very similar to the direct computation of (x_l, y_l) from (3.2) - (3.3), but, in fact, if we know a description of the $N \cap S$-components and/or the expressions for the equations corresponding to (3.8) and (3.9), the calculation costs can be reduced considerably.
From the equations (3.5) - (3.9) it can be deduced that $x_l^+ \in N \cap S$. This follows because (3.5) implies $x_l^+ \in S$ and, if we multiply (3.5) by $PQ_1A_2^{-1}$, we obtain $PQ_1x_l^+ = 0$. Taking into account (3.7), $x_l^+ \in N$ has to be given.
This property will be of special interest with regard to circuit simulation.

Remarks

1. At first glance, this approach seems not to be very helpful, because a value that fulfils the system's equations has to be given a priori. Fortunately, in circuit simulation we can take advantage of the structural properties that guarantee the existence of the DC operating point to compute such a value.

2. The circuit simulation by means of MNA leads to quasi-linear DAEs. The described projectors, spaces and index definitions of the tractability index can be extended to nonlinear systems (cf. [10]). The definitions for the equations arising from circuit simulation by means of modified nodal analysis were discussed in detail in [8]. There it was proved that, under certain restrictions on the controlled sources (see Tables 5.1 and 5.2), the space $N \cap S$ is constant and the $N \cap S$-component appears only in linear relations of the DAE. Therefore, the above approach can be successfully extended.

3. In [6] it was already pointed out how to transcribe topologically the equations that describe the hidden constraints analogously to (3.8) with the aid of constant projectors. Nevertheless, the topological initialization presented there is finally based on the idea of fixing only the dynamic components and calculate the values for the remaining variables by means of the system's equations. This approach has the advantage that no specific (x^l, y^l) has to be given, but the disadvantage that the obtained values depend on the choice of the variables for which initial values have been prescribed.

4. In the course of this article it will be shown that an electrical explanation for the rearrangement from (x^l, y^l) to (x_l, y_l) can be given.

4 The MNA equations

Let us analyze the DAE system obtained by the application of the MNA from lumped networks containing nonlinear and possibly time-variant resistances, capacitances, inductances, independent voltage and current sources, and some specific controlled sources.

We denote by q and ϕ the charge associated with the capacitances and the fluxes associated with the inductances, by j_L and j_V the current vector of inductances and voltage sources and by e the vector of node potentials.

On the other hand, $i(\cdot)$, and $v(\cdot)$ represent functions of current and voltage sources. In this paper, we will assume special prerequisites for the controlled sources.

Analogously to [8], n-terminal resistances, capacitances and inductances are completely described by $(n-1)$ currents entering the $(n-1)$ terminals and then $(n-1)$ branch voltages across each of these $(n-1)$ terminals and the reference terminal n.

To write down the MNA [7] equations, we split the reduced incidence Matrix A into the element-related incidence matrices $A = (A_C A_L A_R A_V A_I)$, where A_C, A_L, A_R, A_V, and A_I describe the branch-current relation for capacitive branches, inductive branches, resistive branches, branches of voltage sources and branches of current sources, respectively.

If we define

$$C(u,t) := \frac{\partial q(u,t)}{\partial u},\ q_t'(u,t) := \frac{\partial q(u,t)}{\partial t},\ L(j,t) := \frac{\partial \phi(j,t)}{\partial j},\ \phi_t'(j,t) := \frac{\partial \phi(j,t)}{\partial t},$$

the DAE system we obtain from networks by the conventional MNA reads

$$\begin{aligned} A_C C(A_C^T e,t) A_C^T \frac{de}{dt} + A_C q_t'(A_C^T e,t) + A_R r(A_R^T e,t) & & \\ + A_L j_L + A_V j_V + A_I i(\cdot) &= 0, & (4.1) \\ L(j_L,t)\frac{dj_L}{dt} + \phi_t'(j_L,t) - A_L^T e &= 0, & (4.2) \\ A_V^T e - v(\cdot) &= 0. & (4.3) \end{aligned}$$

Later on we will also need $G(u,t) := \frac{\partial r(u,t)}{\partial u}$.

We first analyze the network with respect to the conventional MNA and, afterwards, extend the results to the systems obtained by charge-oriented MNA. These systems are [8]:

$$A_C \frac{dq}{dt} + A_R r(A_R^T e,t) + A_L j_L + A_V j_V + A_I i(\cdot) = 0, \qquad (4.4)$$

$$\frac{d\phi}{dt} - A_L^T e = 0, \qquad (4.5)$$

$$A_V^T e - v(\cdot) = 0, \qquad (4.6)$$

$$q - q_C(A_C^T e,t) = 0, \qquad (4.7)$$

$$\phi - \phi_L(j_L,t) = 0. \qquad (4.8)$$

[7] A detailed discussion on how we set up the system's equations can be found in [8] and [12].

[8] cf. again [8] and [12].

Analogously to [8] and [6], we suppose that the capacitance matrix $C(A_C^T e, t)$, inductance matrix $L(j_L, t)$, and conductance matrix $G(A_R^T e, t)$ of all capacitances, inductances and resistances, respectively, are positive definite [9].
We will also make use of the fact that the reduced incidence matrix $(A_C A_L A_R A_V)$ has full row rank and that A_V has full column rank, because cutsets of current sources only and loops of voltage sources only are forbidden (cf. [18], [8]).

5 Index analysis and consistent initialization for circuit simulation

5.1 Some definitions and results

In this section we repeat some of the results presented in [8] and [6] concerning the index of the DAE system and the expressions for the hidden constraints in terms of appropriate projectors. To this end, we need the following definitions and results.

Definition 5.1 *To characterize the topological properties of the network, we define the projectors Q_C, Q_{V-C}, $\bar{Q}_{V-C}$, and Q_{R-CV} onto $\ker A_C^T$, $\ker A_V^T Q_C$, $\ker Q_C^T A_V$, and $\ker A_R^T Q_C Q_{V-C}$, respectively.*
Note that $Q_{CRV} := Q_C Q_{V-C} Q_{R-CV}$ is a projector onto $\ker(A_C A_R A_V)^T$.
The complementary projectors will be denoted by $P_{**} := I - Q_{**}$, with the corresponding subindices.

Definition 5.2

1. *An L-I cutset is a cutset consisting of inductances and/or current sources only.*
2. *A C-V loop is a loop consisting of capacitances and voltage sources.*

In [8] , [18] the following was shown to hold:

Lemma 5.3

1. *If the network does not contain L-I cutsets, then $Q_{CRV} = 0$.*
2. *If the network does not contain C-V loops, then $\bar{Q}_{V-C} = 0$.*

In this article, we suppose that the controlled sources that form part of the network fulfil the conditions exposed below in the Tables 5.1 and 5.2.

Regarding equations (5.3), (5.5), and (5.7) from Table 5.2, the assumptions made for the controlled current sources imply that

$$Q_{CRV}^T A_I i((A_C A_V A_R)^T e, j_L, j_V, t) = Q_{CRV}^T A_{It} i_t \tag{5.9}$$

[9] Of course, the same restriction on the positive definiteness of the conductance matrix from Corollary 2.2 of [8] can be made here. Therefore, for the resistances with incidence nodes that are connected to each other by capacitances and/or voltage sources, no positive definiteness of the corresponding conductance matrix has to be assumed.

If we consider the element-related splitting of $\bar{Q}_{V-C}$, i. e.,

$$\bar{Q}_{V-C} = \begin{pmatrix} (\bar{Q}_{V-C})_t \\ (\bar{Q}_{V-C})_{contr.} \end{pmatrix},$$

then we can summarize the prerequisites we assume for the controlled voltage sources as follows:

$$\begin{aligned} \bar{Q}_{V-C}^T v(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) &= \bar{Q}_{V-C}^T v_t(t), \quad (5.1) \\ v(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) &= v_*(A_C^T e, j_L, t) \quad (5.2) \end{aligned}$$

for a suitable function v_* and for a vector $v_t(t)$ that contains the functions of independent voltage sources and zeros instead of the functions of controlled voltage sources. Analogously as in [8] and [6], in the following we will drop the index *.

Table 5.1: Condition for controlled voltage sources

is always fulfilled. Thus, we generally assume that the controlled sources do not form part of the C-V loops or L-I cutsets.
To shorten denotations we write

$$i((A_C A_V A_R)^T e, j_L, \bar{P}_{V-C} j_V, t) \quad (5.10)$$

when we do not distinguish between (5.4), (5.6), and (5.8).[10]

Lemma 5.4 *The matrices*

$$\begin{aligned} H_1(A_C^T e,t) &:= A_C C(A_C^T e,t) A_C^T + Q_C^T Q_C, \\ H_2 &:= Q_C^T A_V A_V^T Q_C + Q_{V-C}^T Q_{V-C}, \\ H_3 &:= A_V^T Q_C Q_C^T A_V + \bar{Q}_{V-C}^T \bar{Q}_{V-C}, \\ H_4(A_C^T e,t) &:= \bar{Q}_{V-C}^T A_V^T H_1^{-1}(A_C^T e,t) A_V \bar{Q}_{V-C} + \bar{P}_{V-C}^T \bar{P}_{V-C}, \\ H_5(j_L,t) &:= Q_{CRV}^T A_L L^{-1}(j_L,t) A_L^T Q_{CRV} + P_{CRV}^T P_{CRV}, \\ H_6 &:= \bar{Q}_{V-C}^T A_V^T A_V \bar{Q}_{V-C} + \bar{P}_{V-C}^T \bar{P}_{V-C}, \\ H_7 &:= Q_{CRV}^T A_L A_L^T Q_{CRV} + P_{CRV}^T P_{CRV}, \end{aligned}$$

are nonsingular.[11]

[10] These assumptions can be transcribed into topological criteria analogously as it was done in [8]. The result would be similar, with the only difference that now the branch potentials of branches that form part of L-I cutsets would not be allowed to control controlled current sources.
[11] cf. [8], [6].

For controlled current sources we suppose that at least one of the following characterizations holds:
(a)

$$
\begin{aligned}
Q_{CRV}^T A_I i(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) &= Q_{CRV}^T A_{It} i_t, && (5.3)\\
i(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) &= i_a(A_C^T e, A_V^T e, j_L, t) && (5.4)
\end{aligned}
$$

for a suitable function i_a.
(b)

$$
\begin{aligned}
Q_C^T A_{Ib} &= 0, && (5.5)\\
i(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) &= i_b((A_C A_V A_R)^T e, j_L, \bar{P}_{V-C} j_V, t) && (5.6)
\end{aligned}
$$

for a suitable function i_b.
(c)

$$
\begin{aligned}
Q_{V-C}^T Q_C^T A_{Ic} &= 0, && (5.7)\\
i(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) &= i_c((A_C A_V A_R)^T e, j_L, t) && (5.8)
\end{aligned}
$$

for a suitable function i_c.

Table 5.2: Conditions for controlled current sources

In [8] the following result was obtained:

Theorem 5.5 *Consider lumped electric circuits satisfying the assumptions of the Tables 5.1 and 5.2. Then it holds:*

1. *If the network contains neither L-I cutsets nor controlled C-V loops, then the conventional MNA leads to a DAE system with index-1 and the constraints are only the explicit ones:*

$$
\begin{aligned}
Q_C^T \big[A_R r(A_R^T e, t) + A_L j_L + A_V j_V & \\
+ A_{Ia,c} i_{a,c}((A_C A_R A_V)^T e, j_L, t)\big] &= 0, && (5.11)\\
A_V^T e - v(A_C^T e, j_L, t) &= 0. && (5.12)
\end{aligned}
$$

2. *If the network contains L-I cutsets or C-V loops, then the conventional MNA leads to a DAE system with index-2. With regard to the constraints, we distinguish the following three possibilities.*

   (a) If the network does not contain an L-I cutset (but contains controlled C-V loops), then the constraints are the explicit ones, (5.11) and

(5.12), and, additionally, the hidden constraint:

$$\bar{Q}_{V-C}^T A_V^T H_1^{-1}(A_C^T e,t) P_C^T \Big[A_C q_t'(A_C^T e,t) + A_R r(A_R^T e,t) + A_L j_L$$
$$+ A_V j_V + A_I i((A_C A_R A_V)^T e, j_L, \bar{P}_{V-C} j_V, t) \Big] + \bar{Q}_{V-C}^T \frac{dv_t}{dt} = 0. \quad (5.13)$$

(b) *If the network does not contain controlled C-V loops, but contains L-I cutsets, the constraints are the explicit ones, (5.11) and (5.12), and, additionally, the hidden constraint:*

$$Q_{CRV}^T \left(A_L L^{-1}(j_L,t) \left(A_L^T e - \phi_t'(j_L,t) \right) + A_{It} \frac{di_t}{dt} \right) = 0. \quad (5.14)$$

(c) *If the network contains L-I cutsets and C-V loops, then the constraints are the explicit ones, (5.11) and (5.12), and the hidden ones (5.13) and (5.14).*

Remark: In [8] it was proved that the hidden constraint (5.13) resulted from

$$\bar{Q}_{V-C}^T A_V^T \frac{de}{dt} = \bar{Q}_{V-C}^T \frac{dv_t}{dt} \quad (5.15)$$

and that (5.14) arose from

$$Q_{CRV}^T \left(A_L \frac{dj_L}{dt} + A_{It} \frac{di_t}{dt} \right) = 0. \quad (5.16)$$

For the sake of simplicity, we will sometimes drop the arguments of the H matrices in the following and write a dot if they are not constant.
In [6] it was shown that a splitting of the system can be performed in such a way that consistent initial values can be calculated successively as described below.

Corollary 5.6 *Let the values $(P_C e^0, j_L^0)$ be given. If the network contains controlled sources that fulfil the conditions of the Tables 5.1 and 5.2, we can determine consistent initial values for the system (4.1)-(4.3) gradually.*

We split $e_0 = P_C e_0 + Q_C P_{V-C} e_0 + Q_C Q_{V-C} P_{R-CV} e_0 + Q_C Q_{V-C} Q_{R-CV} e_0$ and $j_{V0} = \bar{Q}_{V-C} j_{V0} + \bar{P}_{V-C} j_{V0}$, and obtain the corresponding consistent values from

$$\begin{aligned}
P_C e_0 &:= P_C e^0 + P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T \left(v_t(t_0) - A_V^T P_C e^0 \right), \\
j_{L0} &:= j_L^0 + A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T \left(-A_{I_t} i_t(t_0) - A_L j_L^0 \right), \\
Q_C P_{V-C} e_0 &:= Q_C H_2^{-1} Q_C^T A_V \bar{P}_{V-C}^T \left(-A_V^T P_C e_0 + v(A_C^T e_0, j_{L0}, t_0) \right),
\end{aligned}$$

whereas the value of $Q_C Q_{V-C} P_{R-CV} e_0$ can be obtained by solving the equation

$$\begin{aligned}
P_{R-CV}^T Q_{V-C}^T Q_C^T \Big[& A_R r(A_R^T (P_C + Q_C P_{V-C} + Q_C Q_{V-C} P_{CRV}) e_0) \\
& + A_L j_{L0} + A_{I_a} i_a(A_C^T e_0, A_V^T e_0, j_{L0}, t_0) \Big] = 0.
\end{aligned}$$

Then we calculate

$$\bar{P}_{V-C}j_{V0} \;:=\; -H_3^{-1}A_V^T Q_C P_{V-C}^T Q_C^T \Big[A_R r(A_R^T e_0, t_0) + A_L j_{L0} + A_{I_{a,c}} i_{a,c}((A_C A_V A_R)^T e, j_L, t) \Big].$$

Next, we can determine the remaining values by means of

$$\begin{aligned} Q_{CRV}e_0 \;&:=\; -\left(Q_{CRV}H_5^{-1}(\cdot)Q_{CRV}^T\right)\cdot \\ &\quad \Big(A_L L^{-1}(j_{L0},t_0)A_L^T\left(P_C + Q_C P_{V-C} + Q_C Q_{V-C}P_{R-CV}\right)e_0 \\ &\quad - A_L L^{-1}(j_{L0},t_0)\phi_t'(j_{L0},t_0) + A_{I_t}\frac{di_t}{dt}(t_0)\Big), \\ \bar{Q}_{V-C}j_{V0} \;&:=\; -H_4^{-1}(\cdot)\bar{Q}_{V-C}^T A_V^T H_1^{-1}(\cdot)P_C^T \Big(A_C q_t'(A_C^T e_0,t_0) + A_R r(A_R^T e_0,t_0) \\ &\quad + A_L j_{L_0} + A_V \bar{P}_{V-C}j_{V_0} + A_I i((A_C A_V A_R)^T e_0, j_{L0}, \bar{P}_{V-C}j_{V0}, t_0)\Big) \\ &\quad - H_4^{-1}(\cdot)\bar{Q}_{V-C}^T\frac{dv_t}{dt}(t_0). \end{aligned}$$

If the charge-oriented MNA is considered, we set additionally:

$$\begin{aligned} q_0 \;&:=\; q_C(A_C^T e_0, t_0), \\ \phi_0 \;&:=\; \phi_L(j_{L0}, t_0). \end{aligned}$$

Remarks:

- Note that each time the matrices $H_1^{-1}(\cdot) = H_1^{-1}(A_C^T e_0, t_0)$, $H_4^{-1}(\cdot) = H_4^{-1}(A_C^T e_0, t_0)$ or $H_5^{-1}(\cdot) = H_5^{-1}(j_{L0}, t_0)$ appear, we already know the corresponding values $A_C^T e_0$ or j_{L0} and, therefore, can insert them. On the other hand, the conditions of the Tables 5.1 and 5.2 imply precisely that this holds analogously for the controlled sources.[12].
- Of course, if the network contains no C-V-loops and no L-I cutsets, then the corresponding equations defined with the aid of the projectors $\bar{Q}_{V-C}$ and Q_{CRV}, respectively, do not appear.
- Corollary 5.6 implies that the choice of (e^0, j_L^0, j_V^0) is arbitrary as long as the nonlinear equation that leads to the expression for $Q_C Q_{V-C} P_{R-CV} e_0$ is solvable.

From the results presented in [8] it follows directly that, for the controlled sources we consider here, the space $N \cap S(\cdot)$ is constant and can be described by

$$N \cap S(\cdot) = \text{im } Q_{CRV} \times \{0\} \times \text{im } \bar{Q}_{V-C}$$

[12] Note that the controlled sources we permit do not change the spaces associated with the DAEs and, in this context, imply that we do not need to alter the order, as it was done in [6].

for the conventional MNA and

$$N \cap S(\cdot) = \{0\} \times \{0\} \times \text{im } Q_{CRV} \times \{0\} \times \text{im } \bar{Q}_{V-C}$$

for the charge-oriented MNA. Observe that $Q_{CRV}e$ and $\bar{Q}_{V-C}j_V$ appear only in linear expressions of the equations (4.1) - (4.3) and (4.4) - (4.8).

5.2 Topological analysis of the network

Let us analyze the topology of a given circuit to locate the equations corresponding to (5.15) and (5.16), i. e., to the equations that lead to the hidden constraints[13]. In [6] it was shown that these equations can be obtained directly from the network by making use of the following two procedures. They precisely determine the linearly independent equations that describe the hidden constraints arising from C-V loops and L-I cutsets, respectively.

PROCEDURE 1

1. Search a C-V loop in the given network graph. If no loop is found, then end.

2. Write the equation resulting from the sum of the derivatives of the characteristic equations of the voltage sources contained in the C-V loop, taking into account the orientation of the loop and the reference direction of the considered branches.
   For instance, if the voltage sources $v_1, ..., v_k$ form a part of the C-V loop and we define

   $$\alpha_i := \begin{cases} +1 & \text{if the orientation of the loop coincides with that of } v_i \\ -1 & \text{else,} \end{cases}$$

   then the equation we write in this step is $\sum_{i=1}^{k} \alpha_i((A_V^T e)'_i - v'_i) = 0$.

3. Form a new network graph by deleting the branch of one voltage source that forms a part of the loop, leaving the nodes unchanged.

4. Return to 1, considering the new network graph.

The following procedure starts again from the initial graph.

PROCEDURE 2

1. Search an L-I cutset. If one is found, then pick an arbitrary inductance of this cutset. Realize that we can always find such an inductance because cutsets of current sources only are forbidden. If no cutset is found, then end.

[13] A similar topological analysis of the network can be found in [3].

2. Write a new equation resulting by derivation of the cutset equation arising from 1.
   For instance, if the current sources $i_1, ..., i_k$ and the inductances $j_{L1}, ..., j_{L\tilde{k}}$ form a part of the L-I cutset and we define

$$\alpha_j \quad := \quad \begin{cases} +1 & \text{if the orientation of the cutset coincides with that of } i_j \\ -1 & \text{else,} \end{cases}$$

$$\tilde{\alpha}_j \quad := \quad \begin{cases} +1 & \text{if the orientation of the cutset coincides with that of } j_{Lj} \\ -1 & \text{else,} \end{cases}$$

   then the equation obtained in this step reads $\sum_{j=1}^{k} \alpha_j i'_j + \sum_{i=1}^{\tilde{k}} \tilde{\alpha}_j j'_{Li} = 0$.

3. Delete the chosen inductance from the network contracting its incident nodes.

4. Return to 1, considering the new network graph.

In [6], an extension of these procedures determines how to fix the dynamic components for calculating consistent initial values. The result permits to assign specific values to a selection of variables, and to determine the remaining by making use of the system's equation[14]. Nevertheless, the values obtained in that way depended on the choice of variables selected.

6 Calculating a consistent value starting up from a value fulfilling the system's equations

The approach presented in this section distributes the index-2 property along all affected elements. It results from the approach presented in Section 3 and can be summarized as follows:

Theorem 6.1 *For networks that contain only controlled sources as specified in the Tables 5.1 and 5.2 we obtain consistent initial values starting up from possibly inconsistent ones that fulfil the system's equations in the following way:*

1. *Add additional currents that flow through the C-V-loops as a consequence of the hidden constraints described by PROCEDURE 1 to the values of the currents through the branches that form a part of C-V-loops.*

2. *Add convenient values to the node potentials to fulfil the additional voltage across the L-I cutsets defined by the hidden constraints described by PROCEDURE 2.*

The meaning of this theorem becomes clear in the course of the article. The Theorems 6.5 and 6.6 describe the statement properly.
In this chapter, we first give an overview of our aim, explaining the approach

[14]Therefore, this result is similar to the one obtained in [16].

for a special case, the DC operating point, and then we generalize the results to apply them for arbitrary points. The proofs will be pointed out for the general case only.
In Section 6.4 we illustrate why the class of controlled sources for which this approach holds cannot be extended if no further topological considerations are made.

6.1 An initialization related to the DC operating point

A common way for obtaining values to start the numerical integration in circuit simulation is to calculate the DC operating point. Therefore, we consider this point separately. We calculate the DC operating point by setting the current through the capacitances and the voltages across the inductances equal to zero. Note that for calculating the DC operating point for the charge-oriented MNA, the matrix $\frac{df}{dx}$ has to be nonsingular.
Topologically, this implies that:

1. Cutsets of capacitances and/or current sources are forbidden, i.e., that (A_L, A_R, A_V) has full row rank.
2. Loops of inductances and/or voltage sources are forbidden, i.e., (A_L, A_V) has full column rank.

Furthermore, the non-singularity of $\frac{df}{dx}$ implies assumptions on the resistances and on the controlled sources.

Similar considerations hold for the conventional MNA, whereas $q_t'(\cdot)$ and $\phi_t'(\cdot)$ have to be considered separately . Observe that, if time-dependent capacitors or inductors appear in the network, then the existence of the DC operating point and the non-singularity of the corresponding matrix $\frac{df}{dx}$ are not equivalent for the conventional MNA.

In the index-2 case the DC operating point has not to be consistent. Nevertheless, the values obtained for the DC operating point, (e^0, j_L^0, j_V^0), fulfil Kirchoff's laws, and precisely the equations

$$\begin{aligned} A_R r(A_R^T e^0, t_0) + A_L j_L^0 + A_V j_V^0 + A_I i(P_{CRV} e^0, {j_L}^0, \bar{P}_{V-C} {j_V}^0, t_0) &= 0, && (6.1)\\ -A_L^T e^0 &= 0, && (6.2)\\ A_V^T e^0 - v(A_C^T e^0, {j_L}^0, t_0) &= 0, && (6.3) \end{aligned}$$

for the conventional MNA, and, additionally, the equations

$$q - q_C(A_C^T e^0, t_0) = 0, \tag{6.4}$$

$$\phi - \phi_L(j_L^0, t_0) = 0, \tag{6.5}$$

if the charge-oriented MNA is considered.

Theorem 6.2 *For the conventional MNA we obtain a consistent initialization* $(e_0, j_{L0}, j_{V0}, P_C e_0', j_{L0}')$ *related to the DC operating point* (e^0, j_L^0, j_V^0) *in the following way:*

1. *Set* $P_{CRV}e_0 := P_{CRV}e^0$, $j_{L0} := j_L^0$, $\bar{P}_{V-C}j_{V0} := \bar{P}_{V-C}j_V^0$ [15].
2. *Calculate* $Q_{CRV}e_0$ *making use of the above setting and solving the equation*
$$Q_{CRV}^T\left(A_L L^{-1}(j_{L0},t_0)\left(A_L^T e_0 - \phi_t'(j_{L0},t_0)\right) + A_{I_t}\frac{di_t}{dt}(t_0)\right) = 0. \quad (6.6)$$
3. *Calculate* $\bar{Q}_{V-C}j_{V0}$ *making use of the above setting and solving the equation*
$$\bar{Q}_{V-C}^T A_V^T H_1^{-1}(A_C^T e_0,t_0)P_C^T\Big[A_C q_t'(A_C^T e_0,t_0) + A_R r(A_R^T e_0,t_0) + A_L j_{L0}$$
$$+A_V j_{V0} + A_I i(P_{CRV}e_0, j_{L0}, \bar{P}_{V-C}j_{V0},t_0)\Big] + \bar{Q}_{V-C}^T\frac{dv_t}{dt}(t_0) = 0.(6.7)$$
4. *Calculate the values of the derivatives of the voltages across the capacitances and of the derivatives of the currents through the inductances according to (4.1) and (4.2) .*

For the charge-oriented MNA the result can be directly adapted by calculating $q_0 = q^0$ *and* $\phi_0 = \phi^0$ *with (4.7) and (4.8) and by making use of (4.4) and (4.5) to calculate the derivatives of the charges and fluxes associated with the capacitances and inductances, respectively.*

Proof:
This result is a direct consequence of Corollary 5.6. The equations (6.1)-(6.3) guarantee that
$$\begin{aligned} Q_{CRV}^T[A_L j_L^0 + A_{I_t}i_t(t_0)] &= 0 \text{ and} \\ \bar{Q}_{V-C}^T A_V^T e^0 &= \bar{Q}_{V-C}^T v_t(t_0) \end{aligned}$$
are fulfilled.
Therefore, the results from Corollary 5.6 lead to $P_{CRV}e_0 = P_{CRV}e^0$, $j_{L0} = j_L^0$, $\bar{P}_{V-C}j_{V0} = \bar{P}_{V-C}j_V^0$, and to the equations for $Q_{CRV}e_0$ and $\bar{Q}_{V-C}j_{V0}$ as described.

q.e.d.

Corollary 6.3 *The consistent initial values defined in Theorem 6.2 can be calculated in the following way:*
$$\begin{aligned} e_0 &:= e^0 - H_5^{-1}(j_L^0,t_0)Q_{CRV}^T\left(A_{I_t}\frac{di_t}{dt}(t_0) - A_L L^{-1}(j_L^0,t_0)\phi_t'(j_L^0,t_0)\right), \\ j_{L0} &:= j_L^0, \\ j_{V0} &:= j_V^0 - H_4^{-1}(A_C^T e^0,t_0)\bar{Q}_{V-C}^T\Bigg(\frac{dv_t}{dt}(t_0) \\ &\quad + A_V^T H_1^{-1}(A_C^T e^0,t_0)A_C q_t'(A_C^T e^0,t_0)\Bigg). \end{aligned}$$

[15] Note that this implies precisely $x^+ \in N \cap S(\cdot)$, (cf. Sections 3 and 5.1).

Then, the corresponding values of the derivatives of voltages across the capacitances and currents through the inductances can be computed by:

$$
\begin{aligned}
P_C e_0' \quad &:= \quad -H_1^{-1}(A_C^T e^0, t_0) A_C q_t'(A_C^T e^0, t_0) \\
&\quad +H_1^{-1}(A_C^T e^0, t_0) A_V H_4^{-1}(A_C^T e^0, t_0) \bar{Q}_{V-C}^T \left(\frac{dv_t}{dt}(t_0) \right. \\
&\quad \left. +A_V^T H_1^{-1}(A_C^T e^0, t_0) A_C q_t'(A_C^T e^0, t_0) \right), \\
j_{L0}' \quad &:= \quad -L^{-1}(j_L{}^0, t_0)\phi_t'(j_L{}^0, t_0) - L^{-1}(j_L{}^0, t_0) A_L^T H_5^{-1}(j_L{}^0, t_0) Q_{CRV}^T \left(A_{I_t} \frac{di_t}{dt}(t_0) \right. \\
&\quad \left. -A_L L^{-1}(j_L^0, t_0)\phi_t'(j_L^0, t_0) \right).
\end{aligned}
$$

For the charge-oriented MNA we set $q_0 := q^0$ and $\phi_0 := \phi^0$, whereas the corresponding values of the nontrivial derivatives of charges associated with the capacitances and fluxes associated with the inductances can be obtained by:

$$
\begin{aligned}
\bar{P}_C q_0' \quad &:= \quad \bar{H}_C^{-1} A_V H_4^{-1}(A_C^T e^0, t_0) \left[\bar{Q}_{V-C}^T \frac{dv_t}{dt}(t_0) \right. \\
&\quad \left. +\bar{Q}_{V-C}^T A_V^T H_1^{-1}(A_C^T e^0, t_0) A_C q_t'(A_C^T e^0, t_0) \right], \\
\phi_0' \quad &:= \quad -A_L^T H_5^{-1}(j_L{}^0, t_0) \left(Q_{CRV}^T A_{I_t} \frac{di_t}{dt}(t_0) - Q_{CRV}^T A_L L^{-1}(j_L{}^0, t_0)\phi_t'(j_L{}^0, t_0) \right),
\end{aligned}
$$

for $\bar{H}_C := A_C^T A_C + \bar{Q}^T \bar{Q}$.

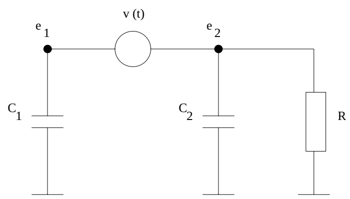
Note that $H_4(\cdot)$ and $H_5(\cdot)$ were defined in Section 5.1. An example is given in Figure 6.1.

Proof: The corollary is a special case of the Theorems 6.5 and 6.6.

Corollary 6.4 *The values obtained in Theorem 6.2 imply that, at time t_0, the sum of the additional power delivered to the network by the C-V-loops and L-I cutsets is equal to the sum of the additional power absorbed by the branches of the C-V-loops and L-I-cutsets.*

Proof: The corollary is a special case of Corollary 6.7.

In Figure 6.2 we can observe how this approach can be carried out for the NAND-Gate described e.g. in [19]. We consider the case that it contains linear capacitances. The result shows that there is a current that flows through V_1, through the MOSFET that is incident with node 6, and through V_{BB}. Note that inside the MOSFET the current is divided.



Conventional MNA:

$$
\begin{aligned}
C_1 e_1' + j_V &= 0, \\
-j_V + C_2 e_2' + \frac{1}{R} e_2 &= 0, \\
e_1 - e_2 &= v(t_0).
\end{aligned}
$$

DC operating point:

$e_1 = v(t_0), \quad e_2 = 0 \quad, j_V = 0.$

Consistent initial value:

$$
\begin{aligned}
e_1 &= v(t_0), \quad e_2 = 0, \\
j_V &= \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} v'(t_0), \\
e_1' &= -\frac{1}{C_1} \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} v'(t_0), \\
e_2' &= \frac{1}{C_2} \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} v'(t_0).
\end{aligned}
$$

Figure 6.1: Circuit with C-V loop

6.2 Computing consistent values starting up from possibly inconsistent ones

For any value (x^l, Py^l) fulfilling the DAE equations but being not necessarily consistent, a consistent (x_l, Py_l) can be computed in an analogous way as for the operating point[16].
Instead of (6.1) - (6.3), (or (6.1) - (6.5), correspondingly) the possibly inconsistent values fulfil:

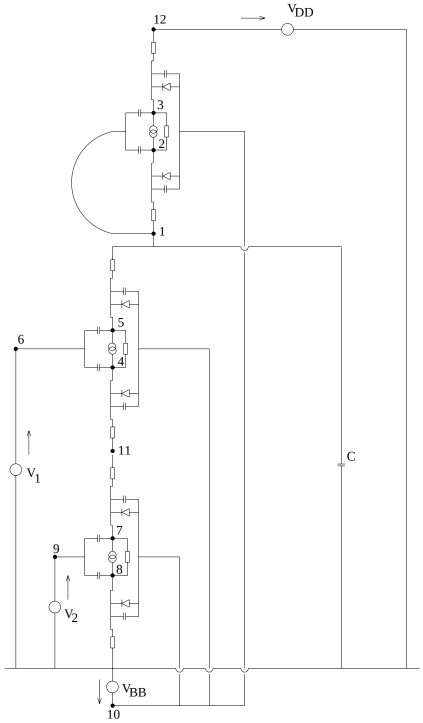
$$
\begin{aligned}
& A_C C(A_C^T e^l, t_l) A_C^T e'^l + A_C q_t'(A_C^T e^l, t_l) + A_R r(A_R^T e^l, t_l) \\
& + A_L j_L^l + A_V j_V^l + A_I i((A_C A_V A_R)^T e^l, j_L{}^l, \bar{P}_{V-C} j_V{}^l, t_l) = 0, \quad (6.8) \\
& L(j_L^l, t_l) j_L'^l + \phi_t'(j_L^l, t_l) - A_L^T e^l = 0, \quad (6.9) \\
& A_V^T e^l - v(A_C^T e^l, j_L{}^l, t_l) = 0, \quad (6.10)
\end{aligned}
$$

for the conventional MNA and

$$
\begin{aligned}
& A_C q'^l + A_R r(A_R^T e^l, t_l) + A_L j_L^l + A_V j_V^l \\
& + A_I i((A_C A_V A_R)^T e^l, j_L{}^l, \bar{P}_{V-C} j_V{}^l, t_l) i = 0, \quad (6.11) \\
& \phi'^l - A_L^T e^l = 0, \quad (6.12) \\
& A_V^T e^l - v(A_C^T e^l, j_L{}^l, t_l) = 0, \quad (6.13) \\
& q^l - q_C(A_C^T e^l, t_l) = 0, \quad (6.14) \\
& \phi^l - \phi_L(j_L^l, t_l) = 0, \quad (6.15)
\end{aligned}
$$

for the charge-oriented MNA.
Consequently, for these values Corollary 5.6 implies, analogously as in Theorem

[16] We introduce the index l to distinguish the values at an arbitrary time t_l from t_0, which was introduced to start the integration process. The results presented in this chapter may be of special interest to calculate consistent values starting up from the possibly inconsistent values for the solution an integration method supplies at t_l.



$$Q_C = \begin{pmatrix} 0 & \dots & & & & 0 \\ : & 0 & & & & : \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & \\ & & & & 1 & 0 \\ 0 & \dots & & & 0 & 1 \end{pmatrix}, \quad \bar{Q}_{V-C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix}
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c0
\end{pmatrix}.$$

The additional current for correcting the DC operating point is:

$$\begin{pmatrix} j^+_{V1} \\ j^+_{V2} \\ j^+_{BB} \\ j^+_{DD} \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{2c0c1}{c0+c1} & 0 & \frac{2c0c1}{c0+c1} & 0 \\ 0 & -\frac{2c0c1}{c0+c1} & \frac{2c0c1}{c0+c1} & 0 \\ \frac{2c0c1}{c0+c1} & \frac{2c0c1}{c0+c1} & -\frac{2c0c1(4c0c1+3c(c0+c1))}{(c0+c1)(2c0c1+c(c0+c1))} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{-H_4^{-1}} \underbrace{\begin{pmatrix} V_1' \\ V_2' \\ V_{BB}' \\ 0 \end{pmatrix}}_{\bar{Q}_{V-C}\frac{dv_t}{dt}}.$$

For $c = 0.5 \cdot 10^{-13}$, $c0 = 0.24 \cdot 10^{-13}$, $c1 = 0.6 \cdot 10^{-13}$, $V_1' = 10^9$, $V_2' = V_{BB}' = 0$ we obtain:

$$\begin{pmatrix} j^+_{V1\,0} \\ j^+_{V2\,0} \\ j^+_{BB\,0} \\ j^+_{DD\,0} \end{pmatrix} = \begin{pmatrix} -3.42857142857142793 \cdot 10^{-5} \\ 0 \\ 3.42857142857142793 \cdot 10^{-5} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} {q'}_0 \\ q'_{1gd\,0} \\ q'_{1gs\,0} \\ q'_{1db\,0} \\ q'_{1sb\,0} \\ q'_{2gd\,0} \\ q'_{2gs\,0} \\ q'_{2db\,0} \\ q'_{2sb\,0} \\ q'_{3gd\,0} \\ q'_{3gs\,0} \\ q'_{3db\,0} \\ q'_{3sb\,0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.71428571428571396 \cdot 10^{-5} \\ 1.71428571428571396 \cdot 10^{-5} \\ 1.71428571428571396 \cdot 10^{-5} \\ 1.71428571428571396 \cdot 10^{-5} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Figure 6.2: Computation of j^+_{V0} and ${q'}^+_0$ for the NAND-Gate from [19].

6.2, that only the values of $Q_{CRV}e^l$ and $\bar{Q}_{V-C}j_V{}^l$ have to be properly changed to obtain consistent values $(e_l, j_{L\,l}, j_{V\,l})$. As a consequence, the values of the appearing derivatives have to be adapted to get consistent (x_l, Py_l) that fulfils the systems corresponding to (4.1) - (4.2), or (4.4) - (4.5).

Theorem 6.5 *For the conventional MNA we obtain consistent values* $(e_l, j_{L\,l}, j_{V\,l})$ *starting up from the possibly inconsistent values* $(e^l, j_L^l, j_V^l, P_C e'^l, j_L'^l)$ *that fulfil the DAE equations as follows:*

$$\begin{aligned}
e_l &:= e^l - H_5^{-1}(j_L^l, t_l) Q_{CRV}^T \left(A_{I_t} \frac{di_t}{dt}(t_l) + A_L j_L'^l \right), \\
j_{L\,l} &:= j_L^l, \\
j_{V\,l} &:= j_V^l - H_4^{-1}(A_C^T e^l, t_l) \bar{Q}_{V-C}^T \left(\frac{dv_t}{dt}(t_l) - A_V^T e'^l \right).
\end{aligned}$$

Furthermore, the corresponding values of the derivatives can be calculated by means of

$$\begin{aligned}
P_C e_l' &:= P_C e'^l + H_1^{-1}(A_C^T e^l, t_l) A_V H_4^{-1}(A_C^T e^l, t_l) \bar{Q}_{V-C}^T \left(\frac{dv_t}{dt}(t_l) - A_V^T e'^l \right), \\
j_{L\,l}' &:= j_L'^l - L^{-1}(j_L^l, t_l) A_L^T H_5^{-1}(j_L^l, t_l) Q_{CRV}^T \left(A_{I_t} \frac{di_t}{dt}(t_l) + A_L j_L'^l \right).
\end{aligned}$$

Proof:
If, for a fixed l, we define

$$\begin{aligned}
e_l^+ &:= e_l - e^l, \\
j_{V\,l}^+ &:= j_{V\,l} - j_V^l,
\end{aligned}$$

then it holds that $e_l^+ = Q_{CRV} e_l^+$ and $j_{V\,l}^+ = \bar{Q}_{V-C} j_{V\,l}^+$, because of $P_{CRV} e_l = P_{CRV} e^l$ and $\bar{P}_{V-C} j_{V\,l} = \bar{P}_{V-C} j_V^l$.
If we insert $e_l = e^l + e_l^+$ and $j_{V\,l} = j_V^l + j_{V\,l}^+$ into the hidden constraints (5.13) and (5.14), which precisely have to be fulfilled additionally by a consistent value, we obtain

$$\begin{aligned}
Q_{CRV}^T \left(A_L L^{-1}(j_{L\,l}, t_l) A_L^T Q_{CRV} e_l^+ + A_L j_L'^l + A_{I_t} \frac{di_t}{dt}(t_l) \right) &= 0 \\
\text{and} \quad \bar{Q}_{V-C}^T \left(A_V^T H_1^{-1}(A_C^T e_l, t_l) A_V \bar{Q}_{V-C} j_{V\,l}^+ - A_V^T P_C e'^l + \frac{dv_l}{dt}(t_l) \right) &= 0.
\end{aligned}$$

The expressions for e_l and $j_{V\,l}$ follow then by multiplying these expressions by $H_5^{-1}(\cdot)$ and $H_4^{-1}(\cdot)$ to calculate e_l^+ and $j_{V\,l}^+$.
Defining now

$$\begin{aligned}
P_C e_l'^+ &:= P_C e_l' - P_C e'^l, \\
j_{L\,l}'^+ &:= j_{L\,l}' - j_L'^l,
\end{aligned}$$

and making use of the fact that $(e_l, j_{L\,l}, j_{V\,l}, P_C e'_l, j'_{L\,l})$ has also to fulfil the system (4.1) - (4.2), we obtain

$$\begin{aligned} A_C C(A_C^T e_l, t_l) A_C^T e_l^{'+} + A_V j_{V\,l}^{+} &= 0, \\ L(j_{L\,l}, t_l) j_L^{'+} - A_L^T e_l^{+} &= 0, \end{aligned}$$

which leads to the presented expressions.

q.e.d.

Theorem 6.6 *For the charge-oriented MNA we obtain consistent values $(e_l, j_{L\,l}, j_{V\,l}, q_l, \phi_l)$ starting up from the possibly inconsistent values $(e^l, j_L^l, j_V^l, q^l, \phi^l, \bar{P}_C q^{'l}, \phi^{'l})$ that fulfil the DAE equations as follows:*

$$\begin{aligned} e_l &:= e^l - H_5^{-1}(j_L^l, t_l) Q_{CRV}^T \left(A_{I_t} \frac{di_t}{dt}(t_l) + A_L L^{-1}(j_L^l, t_l) \left(\phi^{'l} - \phi'_t(j_L^l, t_l) \right) \right), \\ j_{L\,l} &:= j_L^l, \\ j_{V\,l} &:= j_V^l - H_4^{-1}(A_C^T e^l, t_l) \bar{Q}_{V-C}^T \left(\frac{dv_t}{dt}(t_l) \right. \\ &\qquad \left. - A_V^T H_1^{-1}(A_C^T e^l, t_l) A_C \left(q^{'l} - q'_t(A_C^T e^l, t_l) \right) \right), \\ q_l &:= q^l, \\ \phi_l &:= \phi^l. \end{aligned}$$

Furthermore, the values of the derivatives of the charges and the fluxes associated with the capacitances and the inductances, respectively, are determined by

$$\begin{aligned} \bar{P}_C q'_l &:= \bar{P}_C q^{'l} + \bar{H}_C^{-1} A_V H_4^{-1}(A_C^T e^l, t_l) \bar{Q}_{V-C}^T \left(\frac{dv}{dt}(t_l) \right. \\ &\qquad \left. - A_V^T H_1^{-1}(A_C^T e^l, t_l) A_C \left(\bar{P}_C q^{'l} - q'_t(A_C^T e^l, t_l) \right) \right), \\ \phi'_l &:= \phi^{'l} - A_L^T H_5^{-1}(j_L^l, t_l) Q_{CRV}^T \left(A_{I_t} \frac{di_t}{dt}(t_l) \right. \\ &\qquad \left. + A_L L^{-1}(j_L^l, t_l) \left(\phi^{'l} - \phi'_t(j_L^l, t_l) \right) \right) \end{aligned}$$

for $\bar{H}_C = A_C^T A_C + \bar{Q}_C^T \bar{Q}_C$.

Proof: Theorem 6.6 follows similarly to Theorem 6.5.

Proof of Corollary 6.3: Corollary 6.3 can be interpreted as a special case of the Theorems 6.5 and 6.6. For $k = 0$ and

$$\begin{aligned} P_C e^{'0} &= -H_1^{-1}(A_C^T e^0, t_0) A_C q'_t(A_C^T e^0, t_0), \\ j_L^{'0} &= -L^{-1}(j_L^0, t_0) \phi'_t(j_L^0, t_0), \end{aligned}$$

or, if the charge-oriented MNA is considered, $q^{'0} = 0$ and $\phi^{'0} = 0$.

q.e.d.

Corollary 6.7 *The values obtained in Theorem 6.5 (or 6.6, correspondingly) imply that, at time t_l, the sum of the additional power delivered by the C-V loops and L-I cutsets is equal to the sum of the additional power absorbed by the branches of the C-V loops and L-I cutsets.*

Proof:
First of all, let us consider the elements that are affected by the corrections of the values. Taking into account $j_{Vl}^{+} = \bar{Q}_{V-C} j_{Vl}^{+}$, it follows from the projector analysis made in [6] that only the currents through those voltage sources that form a part of C-V loops are affected. Considering now that the expressions of Theorem 6.6 imply

$$A_C {q'}_l^{+} + A_V j_{Vl}^{+} = 0$$

for $q_l^{'+} = q_l' - q'^l$, it follows that only the currents through capacitances that form a part of C-V loops change.

Taking into account that Kirchoff's laws are valid for the (x^l, Py^l) and for (x_l, Py_l) because both fulfil the MNA equations, Tellegen's theorem[17] implies

$$\sum_{k=1}^{b} u_{l\,k} j_{l\,k} = 0 \quad \text{and} \quad \sum_{k=1}^{b} u_k^l j_k^l = 0, \qquad \text{i.e.,} \quad 0 = \sum_{k=1}^{b} u_{l\,k} j_{l\,k} - \sum_{k=1}^{b} u_k^l j_k^l,$$

if b is the number of branches, $u_{l\,k}$ and $j_{l\,k}$ are the voltages and currents obtained for the network from the consistent value (x_l, Py_l), and u_k^l and j_k^l are the voltages and currents obtained from (x^l, Py^l).
Setting $u_l^{+} := e_l - e^l$, $j_V{}_l^{+} := j_{V\,l} - j_V^l$, $P_C e_l^{'+} := P_C e_l' - P_C e^{'l}$, and $j_L{}_l^{'+} := j_L{}_l' - j_L^{'l}$ for a fixed l, for the conventional MNA this leads to

$$\begin{aligned} 0 \;=\; & \sum_{\substack{L's\ from \\ L-I-cutsets}} \left(L(j_{L\,l}, t_l) j_{lk}^{'+} \right) j_k^l \;+\; \sum_{\substack{C's\ from \\ C-V-loops}} u_k^l \left(C(A_C^T e_l, t_l) A_C^T e_{lk}^{'+} \right) \\ & + \sum_{\substack{V's\ from \\ C-V-loops}} v_{t\,k}(t_l) j_{V\,l}{}_k^{+} \;+\; \sum_{\substack{I's\ from \\ L-I-cutsets}} u_l{}_k^{+} i_{t\,k}(t_l). \end{aligned}$$

and, if $q_l^{'+} := q_l' - q^{'l}$ and $\phi_l^{'+} := \phi_l' - \phi^{'l}$ is defined, the charge-oriented MNA implies

$$\begin{aligned} 0 \;=\; & \sum_{\substack{L's\ from \\ L-I-cutsets}} \phi_{lk}^{'+} j_k^l \;+\; \sum_{\substack{C's\ from \\ C-V-loops}} u_k^l q_{lk}^{'+} \\ & + \sum_{\substack{V's\ from \\ C-V-loops}} v_{t\,k}(t_l) j_{V\,l}{}_k^{+} \;+\; \sum_{\substack{I's\ from \\ L-I-cutsets}} u_l{}_k^{+} i_{t\,k}(t_l). \end{aligned}$$

[17] cf. for instance [5].

q.e.d.

Proof of Corollary 6.4: Corollary 6.4 can be interpreted as a special case of Corollary 6.7 for $k = 0$.

Remark: If x^l denotes a value that fulfils the equations of the system and x_l the corresponding consistent value, then the above results imply that, if we start the integration process with an implicit Euler method, then we will obtain the same results when starting up from x^l and from x_l. This is due to the fact that, on the one hand, the dynamic components of x^l and x_l coincide, and, on the other hand, the same Jacobian for the Newton method results in both cases, because of the linear occurrence of the correction. Observe that this is not the case for integration methods that utilize the values of all variables from the preceding steps, as for instance the trapezoidal rule.

6.3 The relevant linear system to calculate the consistent values

The expressions from the Sections 6.1 and 6.2 can be transformed in a way that we only have to solve a relatively small linear system. We present the results for an arbitrary point.

Recall that we have denoted the n-terminal capacitances, inductances and resistances by capacitances, inductances and resistances . In this way, we do not have to distinguish if some of them control others or are controlled by others. In the following our aim is to define equations that permit the calculation of the values defined in the last section, but that do not require the direct calculation of the inverses of the complete matrices $H_1(\cdot)$, $H_4(\cdot)$, $L(\cdot)$, and $H_5(\cdot)$. Note that these matrices were constructed by complementing the relevant matrices that described the C-V loops and L-I cutsets to obtain the non-singularity. In practice, considerably smaller matrices can be considered.

Definition 6.8 *Denote by A_{C*} and A_{V*} the incidence matrices of the capacitances and the voltage sources that form a part of C-V loops, respectively, and denote by q^* and $C^*(A_{C*}^T e, t)$ the charges and the capacitance matrix corresponding to them.*
Further let A_{L*} denote the incidence matrix of the inductances that form a part of L-I cutsets, and ϕ^*, j_{L*} and $L^*(j_{L*}, t)$ the fluxes, currents and the inductance matrix corresponding to them.

Theorem 6.9 *For the conventional MNA the solution $(\bar{Q}_{V*-C*} {j_{V*}}_l^+, P_{C*} {e_l^{'}}^+)$ of the linear system:*

$$A_{C*} C^*(A_{C*}^T e^l, t_l) A_{C*}^T {e_l^{'}}^+ + A_{V*} \bar{Q}_{V*-C*} {j_{V*}}_l^+ = 0, \qquad (6.16)$$

$$\bar{Q}_{V*-C*}^T A_{V*}^T {e_l^{'}}^+ + \bar{Q}_{V*-C*}^T A_{V*}^T {e'}^l - \bar{Q}_{V*-C*}^T {v_*}_t'(t_l) = 0 \qquad (6.17)$$

and the solution $(Q_{CRV}e_l^+\ ,\ j_{L*}{}_l^{'+})$ *of the linear system*

$$\begin{aligned} L^*(j^l_{L*},t_l)j_{L*}{}_l^{'+} - A^T_{L*}Q_{CRV}e_l^+ &= 0, \qquad (6.18)\\ Q^T_{CRV}A_{L*}j_{L*}{}_l^{'+} + Q^T_{CRV}A_{L*}j_{L*}{}^{'l} + Q_{CRV}A_{I_t}i'_t(t_l) &= 0 \qquad (6.19)\end{aligned}$$

provide the values that permit us to calculate the consistent values from Theorem 6.5.
Note that $P_{C*}e^{'l}$ *and* $j_{L*}{}^{'l}$ *are considered to be constant vectors.*

Remarks:

1. Equations (6.17) and (6.19) can be obtained by making use of Procedures 1 and 2 from Section 5.2.

2. A practicable realization of the calculation of suitable values can be carried out making use of the projectors $\bar{Q}_{V-C}$ and Q_{CRV} defined in [6].

Theorem 6.10 *For the charge-oriented MNA the solution of the linear system*[18]

$$\begin{aligned} A_{C*}C^*(A^T_{C*}e^l,t_l)A^T_{C*}e_l^{'+} + A_{V*}\bar{Q}_{V*-C*}j_{V*}{}_l^+ &= 0, \qquad (6.20)\\ \bar{Q}^T_{V*-C*}A^T_{V*}e_l^{'+} + \bar{Q}^T_{V*-C*}A^T_{V*}e^{'l} - \bar{Q}^T_{V*-C*}v_*{}'_t(t_l) &= 0, \qquad (6.21)\\ q^{'l*} - C^*(A^T_{C*}e^l,t_l)A^T_{C*}e^{'l} - q^*{}'_t(A^T_{C*}e^l,t_l) &= 0 \qquad (6.22)\end{aligned}$$

and the solution of the linear system

$$\begin{aligned} L^*(j^l_{L*},t_l)j^l_{L*l}{}^{'+} - A^T_{L*}Q_{CRV}e_l^+ &= 0, \qquad (6.23)\\ Q^T_{CRV}A_{L*}j_{L*}{}_l^{'+} + Q^T_{CRV}A_{L*}j_{L*}{}^{'l} + Q_{CRV}A_Ii'_t(t_l) &= 0, \qquad (6.24)\\ \phi^{'l*} - L^*(j_{L*}{}^l,t_l)j^{'+}_{L*l} - \phi^*{}'_t(j_{L*}{}^l,t_l) &= 0 \qquad (6.25)\end{aligned}$$

provide the values required to calculate the consistent values e_l *and* $j_{V\,l}$ *from Theorem 6.6. The corresponding values of the concerned derivatives can be fixed then by:*

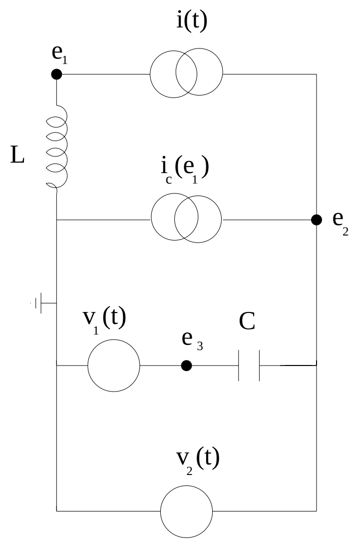
$$\begin{aligned} q_l^{'*} &= q^{'l*} + C^*(A^T_{C*}e^l,t_l)A^T_{C*}e_l^{'+},\\ \phi_l^{'*} &= \phi^{'l*} + L^*(j_{L*}{}^l,t_l)j^{'+}_{L*l} \quad . \end{aligned}$$

Note that now $q^{'l*}$ *and* $\phi^{'l*}$ *are considered to be constant vectors.*

Proof: The theorems follow by straight-forward computation.

Remark: The remarks we made for the conventional MNA hold for the charge-oriented MNA analogously.

[18] Note that we expand the system considered in Theorem 6.9 to avoid the calculation of the inverse matrices to establish the relation between the given values $q^{'l*}$, $\phi^{'l*}$ and the values $\bar{Q}^T_{V-C}A^T_{V*}e^{'l}$, $j_{L*}{}^{'l}$ that appear in (6.21) and (6.24), respectively.



DC operating point:

$$
\begin{aligned}
e_1 &= 0, \;\; e_2 = v_2(t_0)\ ,\ e_3 = v_1(t_0), \\
j_{V1} &= 0\ ,\ j_{V2} = i(t_0) - i_c(0)\ ,\ j_L = -i(t_0), \\
j_L' &= 0\ ,\ C(e_2' - e_3') = 0
\end{aligned}
$$

Consistent initial value:

$$
\begin{aligned}
e_1 &= -Li'(t_0), \;\; e_2 = v_2(t_0), \;\; e_3 = v_1(t_0) \\
j_{V1} &= C(v_2'(t_0) - v_1'(t_0))\ , \\
j_{V2} &= i(t_0) - i_c(-Li'(t_0)) - C(v_2'(t_0) - v_1'(t_0))\ , \\
j_L &= -i(t_0), \\
j_L' &= -i'(t_0)\ ,\ C(e_2' - e_3') = C(v_2'(t_0) - v_1'(t_0)).
\end{aligned}
$$

Figure 6.3: Circuit with VCCS

6.4 Considering more general controlled sources

For arbitrary controlled sources, the index may change in various ways[19]. Therefore, we restrict our considerations to those presented in [8] to notice that the results of the latter section cannot even be applied to all sources considered there. As it was proved in that paper, some of the controlling sources change the structure of the spaces associated with the DAE-system. Indeed, it was proved there that the alteration of the order in which we solve the equations in Theorem 3.2 of [8] became also recognizable in terms of the space $N \cap S(\cdot)$ of the tractability index. For the approach presented in the above section, we use that this space is constant. Furthermore, we utilize that precisely the $N \cap S(\cdot)$-components occur only linearly in the equations (4.1) -(4.3) (or (4.4) -(4.6), respectively) and that, if we modify their values, this changes only the values of the derivatives of the capacitances and inductances that form a part of C-V loops or L-I cutsets.

Unless these assumptions are fulfilled, the presented approach to calculate a consistent value starting from a value fulfilling the system's equations fails. The example of Figure 6.3 illustrates the problem for a class of sources described in [8]. Note that, if, in Figure 6.3, we replace v_2 by a resistance, then, even if there appears no C-V loops, the current through v_1 is affected by a hidden constraint. Therefore, j_{V1} and $C(e_2' - e_3')$ have to be rearranged, too.

Nevertheless, the sources described in the Tables 5.1 and 5.2 include, for instance, the controlled sources contained in the MOSFET.

[19]cf. for instance [13] and [17].

7 A combined possibility of initialization

In the following, we present an initialization approach that may allow the user of a simulation package to prescribe the values of some voltages across capacitances and currents through inductances, and then to calculate an initial value that is still related to a DC operating point. This approach is only possible if the selected elements fulfil some topological restriction described below. In this chapter we will present the approach successively and summarize the result finally.

Recall that for multi-terminal elements with n terminals, we consider each pair of terminals l, n, with $1 \leq l \leq n-1$ as branches, if n denotes the reference terminal[20]. In the following, we assume that, if n-terminal capacitances and inductances appear in the network, then for each one the user either assigns values to all or to none of its branch potentials and currents, respectively. [21]

Denotation 7.1 *Suppose that the user wants to prescribe the values of the branch voltages across some capacitive branches and the values of the currents through some inductive branches of a network G.*
Denote by $\hat{G}$ the graph of the network we obtain when substituting, in G, all capacitances and inductances for which the user wants to prescribe an initial value by independent constant voltage and current sources, respectively. Let the constant values characterizing these sources be precisely the values the user wants to assign.
If the user wants to prescribe the values of an n-terminal, then, to construct $\hat{G}$, we introduce voltage or current sources that connect the reference node of that n-terminal with each of the other terminals.

Let the matrices $(A_C A_R A_L A_V A_I)$ and $(A_{\hat{C}} A_{\hat{R}} A_{\hat{L}} A_{\hat{V}} A_{\hat{I}})$ denote the incidence matrices of the original and the modified graphs, respectively.

We denote by:

- A_{C^-} the incidence matrix of those capacitances across which the user wants to assign an initial branch voltage.
- A_{L^-} the incidence matrix of those inductances through which the user wants to assign a initial current.
- Note that then $A_{\hat{C}}$ and $A_{\hat{L}}$ denote the incidence matrices of capacitances and inductances for which the user does not assign an initial value.
- $A_{\hat{V}+}$ the incidence matrix of the voltage sources we introduce into the network instead of capacitances.
- $A_{\hat{I}+}$ the incidence matrix of the current sources we introduce into the network instead of inductances.

[20] cf. [8].

[21] If this restriction is not kept, then further topological considerations concerning controlling and the controlled elements have to be made.

With these denotations, the following relations follow directly if we suppose that the incidence matrices are defined considering the elements of the same shape successively:

$$A_{C^-} = A_{\hat{V}^+}, \qquad A_{L^-} = A_{\hat{I}^+},$$
$$A_{\hat{V}} = (A_V A_{\hat{V}^+}) = (A_V A_{C^-}), \qquad A_{\hat{I}} = (A_I A_{\hat{I}^+}) = (A_I A_{L^-}),$$
$$A_C = (A_{\hat{C}} A_{C^-}) = (A_{\hat{C}} A_{\hat{V}^+}), \qquad A_L = (A_{\hat{L}} A_{L^-}) = (A_{\hat{L}} A_{\hat{I}^+}).$$

To apply the approach presented in this section, the choice of capacitances and inductances made has to fulfil the topological restrictions of Table 7.1.

TOPOLOGICAL RESTRICTIONS

In this section, a specific choice of inductances and capacitances for which the user wants to prescribe initial values has to fulfil the following topological conditions:

1. The matrix $(A_V A_{C^-}) = A_{\hat{V}}$ has full column rank.
2. The matrix $(A_C A_{L^-} A_R A_V) = (A_{\hat{C}} A_{\hat{R}} A_{\hat{L}} A_{\hat{V}})$ has full row rank.
3. The matrix $(A_{L^-} A_V A_{C^-}) = (A_{\hat{L}}, A_{\hat{V}})$ has full column rank.
4. The matrix $(A_{L^-} A_R A_V A_{C^-}) = (A_{\hat{R}}, A_{\hat{L}}, A_{\hat{V}})$ has full row rank.

If these conditions are fulfilled, we go on looking at $\hat{G}$.

Table 7.1: Topological restrictions

Remarks about Table 7.1:

- Note that the conditions (1) - (2) mean that $\hat{G}$ does not contain loops of voltage sources only nor cutsets of current sources only.
  If these conditions are not fulfilled, then the initialization the user wants to perform is not possible. On the one hand, if (1) is not fulfilled, then it is not possible to assign a value to all chosen capacitances. On the other hand, if (2) is not fulfilled, we cannot prescribe values for all chosen inductances. (cf. [6]).
- Furthermore, the conditions (3) - (4) have to be fulfilled if $\hat{G}$ is DC-solvable. If these conditions are not met, this approach cannot be carried out for the selected inductances and capacitances. In this case, some further or fewer inductances and/or capacitances have to be considered.
- Indeed, these conditions are necessary but not sufficient to guarantee the DC-solvability of $\hat{G}$. For the approach of this section we require that $\hat{G}$ is DC solvable.

From now on, let us suppose that $\hat{G}$ is DC solvable and that we calculate the DC operating point for $\hat{G}$.

In the following we will show how and why we can make use of the values computed in this way to obtain a value that fulfils the MNA equations corresponding to G.

We denote by

- $(\hat{q}, \hat{\phi}, (A_{\hat{C}} A_{\hat{R}} A_{\hat{L}} A_{\hat{V}})^T e, j_{\hat{L}}, j_{\hat{V}})$ the variables corresponding to $\hat{G}$;
- $v_{\hat{V}+}$ and $i_{\hat{I}+}$ the voltage and the current of the introduced independent constant voltage and current sources, respectively;
- $\hat{v}$ and $\hat{i}$ the complete voltage and current vectors of the voltage and current sources of $\hat{G}$.

Making use of the equations that are fulfiled for the DC operating point of $\hat{G}$, we define initial values for G by the following possible relations:

$$
\begin{aligned}
A_{\hat{R}} r(A_{\hat{R}}^T e, t) + A_{\hat{L}} j_{\hat{L}} + \underbrace{A_{\hat{V}} j_{\hat{V}}}_{=A_V j_V + A_{\hat{V}+} j_{\hat{V}+}} + \underbrace{A_{\hat{I}} \hat{i}(\cdot)}_{=A_I i(\cdot) + A_{\hat{I}+} i_{\hat{I}+}} &= 0, && (7.1)\\
-A_{\hat{L}}^T e &= 0, && (7.2)\\
\underbrace{A_{\hat{V}}^T e - \hat{v}(\cdot)}_{=\ A_V^T e - v(\cdot) = 0,\ A_{V+}^T e - v_{V+} = 0} &= 0, && (7.3)\\
\hat{q} - \hat{q}_{\hat{C}}(A_{\hat{C}}^T e, t) &= 0, && (7.4)\\
\hat{\phi} - \hat{\phi}_{\hat{L}}(j_{\hat{L}}, t) &= 0. && (7.5)
\end{aligned}
$$

Of course, if the conventional MNA is considered, the equations (7.4) - (7.5) will not appear.

Assignment 7.2 *Denote by $(\hat{q}^0, \hat{\phi}^0, (A_{\hat{C}} A_{\hat{R}} A_{\hat{L}} A_{\hat{V}})^T e^0, j_{\hat{L}}^0, j_{\hat{V}}^0)$ the operating point of $\hat{G}$. Let us fix then the following values for the variables of G by means of:*

$$
\begin{aligned}
A_C^T e^l &:= \begin{pmatrix} A_{\hat{C}}^T e^0 \\ A_{\hat{V}+}^T e^0 \end{pmatrix} = \begin{pmatrix} A_{\hat{C}}^T e^0 \\ v_{\hat{V}+} \end{pmatrix}, && (7.6)\\
j_L^l &:= \begin{pmatrix} j_{\hat{L}}^0 \\ i_{\hat{I}+} \end{pmatrix}. && (7.7)
\end{aligned}
$$

For the remaining values of $A^T e^l$ and for j_V^l we apply the values obtained for $\hat{G}$ directly.
If we denote by j_C^l the vector that contains the values of j_{V+}^0 for the capacitances for which we prescribed initial branch voltages and zero else, then we can set:

$$
\begin{aligned}
P_C {e'}^l &:= C^{-1}(A_C^T e^l, t_l)(j_C^l - q_t'(A_C^T e^l, t_l)), && (7.8)\\
{j'}_L^l &:= L^{-1}(j_L^l, t_l)(A_L^T e^l - \phi_t'(j_L^l, t_l)) && (7.9)
\end{aligned}
$$

for the conventional MNA and

$$
\begin{aligned}
\frac{dq^l}{dt} &:= j_C^l, && (7.10)\\
\frac{d\phi^l}{dt} &:= A_L^T e^l, && (7.11)\\
q^l &:= q_C(A_C^T e^l, t_l), && (7.12)\\
\phi^l &:= \phi_L(j_L^l, t_l) && (7.13)
\end{aligned}
$$

for the charge-oriented MNA .
Note that the derivatives of the charges and fluxes can only be distinct from zero for the capacitances and inductances for which we prescribed an initial value.

Lemma 7.3 *The values determined by Assignment 7.2 fulfil the MNA equations for G.*

Proof: For the charge-oriented MNA the equations

$$A_C \frac{dq}{dt} + A_R r(A_R^T e, t) + A_L j_L + A_V j_V + A_I i(\cdot) = 0, \tag{7.14}$$

$$\frac{d\phi}{dt} - A_L^T e = 0, \tag{7.15}$$

$$A_V^T e - v(\cdot) = 0, \tag{7.16}$$

$$q - q_C(A_C^T e, t) = 0, \tag{7.17}$$

$$\phi - \phi_L(j_L, t) = 0 \tag{7.18}$$

have to be fulfilled. We discuss each of them.

- Equation (7.14) is fulfilled because of (7.1), taking into account that we obtain

  $$
  \begin{aligned}
  A_C \frac{dq^l}{dt} &= A_{\hat{C}} j_C^l = A_{C^-} j_{\hat{V}+}^0 = A_{\hat{V}+} j_{\hat{V}+}^0,\\
  A_L j_L^l &= A_{\hat{L}} j_{\hat{L}}^0 + A_{\hat{I}+} i_{I+},
  \end{aligned}
  $$

  making use of (7.10), (7.7), and $A_R r(A_R^T e^l, t) = A_{\hat{R}} r(A_{\hat{R}}^T e^0, t)$.

- Equation (7.15) is fulfilled because of (7.2) and the setting (7.11).
- Equation (7.16) is fulfilled because of (7.3).
- Equation (7.17) and (7.18) are fulfilled because of (7.12) and (7.13).

Therefore, this value fulfils the system's equations.

q.e.d.

For the conventional MNA, the proof is analogous.

Consider now the following procedure to calculate a consistent initial value that meets the demands of the user of the simulation package.

PROCEDURE 3

Suppose that the user wants to prescribe values for the branch potentials across the capacitances $C_1, ..., C_n$ and values for the currents through the inductances $L_1, ..., L_m$.

1. Check if the graph $\hat{G}$ resulting according to Denotation 7.1 for the selection of capacitances and inductances is DC-solvable. If not, this procedure cannot be applied.

2. Compute the DC operating point for $\hat{G}$.

3. Fix the values for G as described in Assignment 7.2.

4. Compute a consistent value related to the value obtained in Step 3 as described in the Theorems 6.5 and 6.6 for the conventional and the charge-oriented MNA, respectively.

We summarize the results of this section in the following theorem:

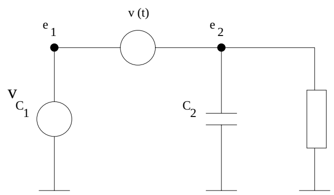
Theorem 7.4 *If a selection of capacitances and inductances is permissible according to step 1, then Procedure 3 yields consistent initial values with the properties:*

- *The branch potentials across the selected capacitances and the currents through the selected inductances are the prescribed ones.*
- *The values of the currents that flow through the capacitances that are neither selected nor form a part of C-V loops are zero.*
- *The values of the voltages across the inductances that are neither selected nor form a part of L-I cutsets are zero.*

An example is given in Figure 7.1. There we consider the example of Figure 6.1 and suppose the user wants to prescribe the value v_{C_1} for the voltage across the capacitance C_1. Observe that the value obtained for the current through the capacitive branch of C_2 depends also on the value v_{C_1}.

8 Conclusion

In this article we have presented how to make use of the special structure of the equations obtained by means of the MNA in electric circuit simulation to compute consistent values. This may give new insight into how to deal with these structural properties with regard to numerical circuit simulation.
Nevertheless, these results are only valid if we make restrictions on the controlled sources that appear in the network. For arbitrary controlled sources, no such general results seem to be possible.



MNA equations for $\hat{G}$:

$$
\begin{aligned}
j_{V\,C_1} + j_V &= 0, \\
-j_V + C_2 e_2' + \frac{1}{R} e_2 &= 0, \\
e_1 - e_2 &= v(t), \\
e_1 &= v_{C_1}.
\end{aligned}
$$

DC operating point for $\hat{G}$:

$$
\begin{aligned}
e_1 &= v_{C_1}, \quad e_2 = v_{C_1} - v(t_0), \\
j_V &= -j_{V\,C_1} = \frac{1}{R}(v_{C_1} - v(t_0)).
\end{aligned}
$$

Definition of values for G:

$$
\begin{aligned}
e_1 &= v_{C_1}, \quad e_2 = v_{C_1} - v(t_0), \\
j_V &= \frac{1}{R}(v_{C_1} - v(t_0)), \\
e_1' &= -\frac{1}{C_1}\frac{1}{R}(v_{C_1} - v(t_0)).
\end{aligned}
$$

These values are not consistent for G.

Consistent initialization for G:

$$
e_1 = v_{C_1}, \quad e_2 = v_{C_1} - v(t_0).
$$

With

$$
j_V^+ = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \left(v'(t_0) - \frac{1}{C_1}\frac{1}{R}(v_{C_1} - v(t_0)) \right)
$$

we obtain

$$
\begin{aligned}
j_V &= \frac{1}{R}(v_{C_1} - v(t_0)) + \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \left(v'(t_0) - \frac{1}{C_1}\frac{1}{R}(v_{C_1} - v(t_0)) \right), \\
e_1' &= -\frac{1}{C_1} \left(\frac{1}{R}(v_{C_1} - v(t_0)) + \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \left(v'(t_0) - \frac{1}{C_1}\frac{1}{R}(v_{C_1} - v(t_0)) \right) \right), \\
e_2' &= \frac{1}{C_2} \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \left(v'(t_0) - \frac{1}{C_1}\frac{1}{R}(v_{C_1} - v(t_0)) \right).
\end{aligned}
$$

Figure 7.1: Example of Procedure 3 if we prescribe the value of the branch potential of the capacitance C_1 in Figure 6.1.

Acknowledgment

I am indebted to C. Tischendorf for valuable comments on a first version of this paper and to U. Feldmann for suggestions that induced these results. I would like to thank R. März, R. Lamour, and S. Sturtzel for many helpful discussions.

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